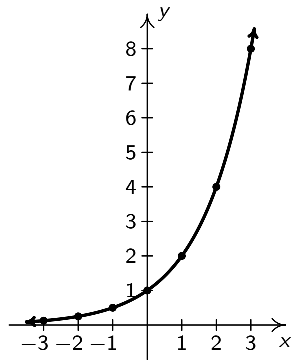
MATH 1650 EXPONENTIAL FUNCTIONS WORKSHEET

EXAMPLE: Fill in the table of values below to graph $f(x) = 2^x$.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$



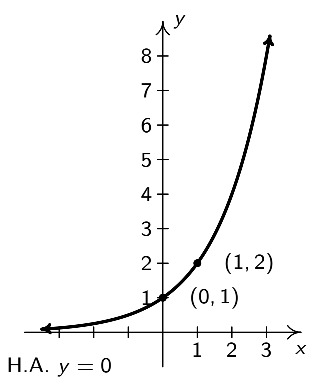
$y = f(x) = 2^x$

What appears to happen as $x \to -\infty$? What appears to happen as $x \to \infty$?

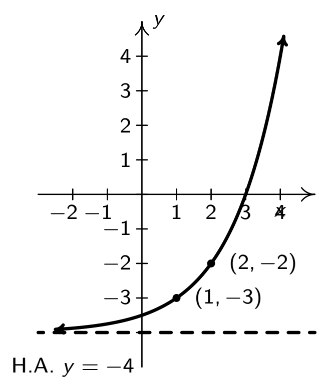
As $x \to -\infty$, $f(x) \to 0$

As $x \to \infty$, $f(x) \to \infty$

EXAMPLE: Graph $f(x) = 2^{x-1} - 4$ by transforming the graph of $y = 2^x$. Track the points $(0, 1)$, $(1, 2)$ and the horizontal asymptote $y = 0$ through the transformations. State the domain and range of f.



$y = 2^x$



$f(x) = 2^{x-1} - 4$

Domain of f: $(-\infty, \infty)$, Range of f: $(-4, \infty)$

EXAMPLE: The value of a car is modeled by $V(t) = 25(0.8)^t$, for $t \geq 0$.

Here, t is number of years the car has been owned and $V(t)$ is the value in thousands of dollars.

- $V(0) = 25(0.8)^0 = 25 \cdot 1 = 25$, $V(1) = 25(0.8)^1 = 25 \cdot 0.8 = 20$ and $V(2) = 25(0.8)^2 = 25 \cdot 0.64 = 16$.

  Since t represents the number of years the car has been owned, $t = 0$ corresponds to the purchase price of the car. Since $V(t)$ returns the value of the car in *thousands* of dollars, $V(0) = 25$ means the car is worth \$25,000 when first purchased.

  Likewise, $V(1) = 20$ and $V(2) = 16$ means the car is worth \$20,000 after one year of ownership and \$16,000 after two years, respectively.

- We compute: $\frac{V(1)}{V(0)} = \frac{20}{25} = 0.8$ and $\frac{V(2)}{V(1)} = \frac{16}{20} = 0.8$.

  The ratio $\frac{V(1)}{V(0)} = 0.8$ can be rewritten as $V(1) = 0.8V(0)$ which means that the value of the car after 1 year, $V(1)$ is 0.8 times, or 80% the initial value of the car, $V(0)$.

  Similarly, the ratio $\frac{V(2)}{V(1)} = 0.8$ rewritten as $V(2) = 0.8V(1)$ means the value of the car after 2 years, $V(2)$ is 0.8 times, or 80% the value of the car after one year, $V(1)$.

  Finally, the ratio $\frac{V(2)}{V(0)} = 0.64$, or $V(2) = 0.64V(0)$ means the value of the car after 2 years, $V(2)$ is 0.64 times, or 64% of the initial value of the car, $V(0)$.

  This last result tracks with the previous answers. Since $V(1) = 0.8V(0)$ and $V(2) = 0.8V(1)$, we get $V(2) = 0.8V(1) = 0.8(0.8V(0)) = 0.64V(0)$.

  Note note it is no coincidence that the base of the exponential, 0.8 has shown up in these calculations, as we'll see in the next problem.

- Using properties of exponents, we find

  $$\frac{V(t+1)}{V(t)} = \frac{25(0.8)^{t+1}}{25(0.8)^t} = (0.8)^{t+1-t} = 0.8$$

  Rewriting, we have $V(t+1) = 0.8V(t)$.

  This means after one year, the value of the car $V(t+1)$ is only 80% of the value it was a year ago, $V(t)$.

  Note that verbally, the function $V(t) = 25(0.8)^t$ says to multiply 25 by 0.8 multiplied by itself t times.

  Therefore, for each additional year, we are multiplying the value of the car by an additional factor of 0.8.

- $\frac{V(1)-V(0)}{V(0)} = \frac{20-25}{25} = -0.2$, $\frac{V(2)-V(1)}{V(1)} = \frac{16-20}{20} = -0.2$, and $\frac{V(2)-V(0)}{V(0)} = \frac{16-25}{25} = -0.36$.

  The ratio $\frac{V(1)-V(0)}{V(0)}$ computes the ratio of *difference* in the value of the car after the first year of ownership, $V(1) - V(0)$, to the initial value, $V(0)$.

  This simplifies to -0.2 or a 20% decrease in value. This makes sense since we know from above that the value of the car after 1 year, $V(1)$ is 80% of the initial value, $V(0)$. Note that:

  $$\frac{V(1) - V(0)}{V(0)} = \frac{V(1)}{V(0)} - \frac{V(0)}{V(0)} = \frac{V(1)}{V(0)} - 1,$$

  and since $\frac{V(1)}{V(0)} = 0.8$, we get $\frac{V(1)-V(0)}{V(0)} = 0.8 - 1 = -0.2$.

  Likewise, the ratio $\frac{V(2)-V(1)}{V(1)} = -0.2$ means the value of the car has lost 20% of its value over the course of the second year of ownership.

  Finally, the ratio $\frac{V(2)-V(0)}{V(0)} = -0.36$ means that over the first two years of ownership, the car value has depreciated 36% of its initial purchase price. Again, this tracks with the result from above which tells us that after two years, the car is only worth 64% of its initial purchase price.

- Using properties of fractions and exponents, we get:

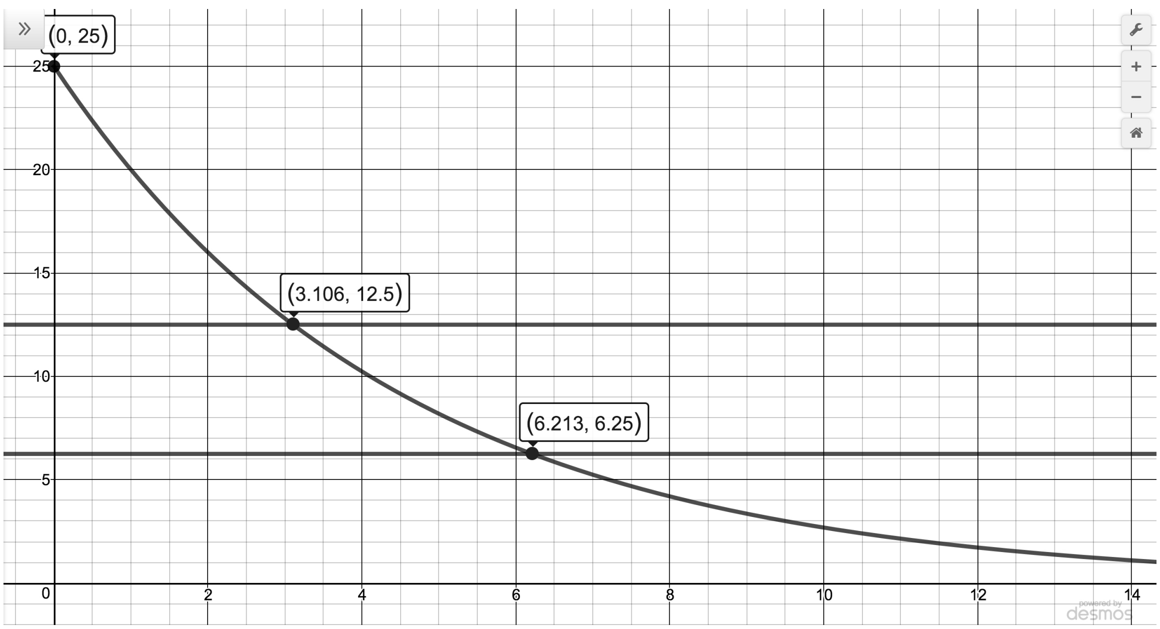
$$\frac{V(t+1)-V(t)}{V(t)} = \frac{25(0.8)^{t+1} - 25(0.8)^t}{25(0.8)^t} = \frac{25(0.8)^{t+1}}{25(0.8)^t} - \frac{25(0.8)^t}{25(0.8)^t} = 0.8 - 1 = -0.2,$$

  so after one year, the value of the car $V(t+1)$ has lost 20% of the value it was a year ago, $V(t)$.

- We know the value of the car, brand new, is \$25,000, so when we are asked to find when the car depreciates to one half and one quarter of this value, we are trying to find when the value of the car dips to \$12,500 and \$6,125, respectively. Since $V(t)$ is measured in *thousands* of dollars, we this translates to solving the equations $V(t) = 12.5$ and $V(t) = 6.125$.

  To solve $V(t) = 12.5$ and $V(t) = 6.125$ graphically, we graph $y = V(t)$ along with the lines $y = 12.5$ and $y = 6.125$ and look for intersection points.

  We find $y = V(t)$ and $y = 12.5$ intersect at (approximately) (3.10612.5) which means the car depreciates to half its initial value in (approximately) 3.11 years. Similarly, we find the car depreciates to one-quarter its initial value after (approximately) 6.23 years.[1]



- We see from the graph of V that its horizontal asymptote is $y = 0$. This means as the car gets older, its value diminishes to 0.

[1] It turns out that it takes exactly twice as long for the car to depreciate to one-quarter of its initial value as it takes to depreciate to half its initial value. Can you see why?